Breakup of Spiral Waves in Coupled Hindmarsh-Rose Neurons

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Breakup of spiral wave in the Hindmarsh–Rose neurons with nearest-neighbour couplings is reported. Appropriate initial values and parameter regions are selected to develop a stable spiral wave and then the Gaussian coloured noise with different intensities and correlation times is imposed on all neurons to study the breakup of spiral wave, respectively. Based on the mean field theory, The statistical factor of synchronization is defined to analyse the evolution of spiral wave. It is found that the stable rotating spiral wave encounters breakup with increasing intensity of gaussian coloured noise or decreasing correlation time to certain threshold.

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Extensive theoretical and experimental studies on firing pattern in neurons have been carried out by establishing some reasonable and biophysical models[1,2] in recent years. Most of the previous works focused on the topics of stochastic resonance[3,4] pattern firing, the functions of neurons, communicating signal among neurons, and synchronization and bifurcation analysis of neurons etc.[5–11] It is such a complex process for receiving and transmitting signal among neurons that many problems are still open to be investigated. As is known, the neuronal system of our body consists of enormous neurons and it is interesting to study the coupling action and pattern transition in the presence of noise so that it can pave a way to know the information encoding in the neurons. More frequently, the interspike intervals (ISIs) of few neurons are easy to be observed and measured but it could be difficult to detect the ISIs of enormous neurons synchronically. Therefore, it is more important to investigate the evolution of spatial firing pattern and find easy way to identify the transition of firing pattern in the spatiotemporal system, which can be described by the coupled oscillators or reaction-diffusion system.

In a theoretical way, spiral wave could be simulated and observed in the reaction-diffusion equation and the coupled oscillators numerically. As we know, spiral wave in two-dimensional spatial space is observed in the physical, biological and chemical systems.[12–15] Especially, the electrical activity in cell[16] can occur spiral wave in the cardiac tissue, which may indicate a kind of heart disease and the breakup of spiral wave just causes a rapid death of heart. There are much similarities between the spiral wave in the excitable media and the cardiac tissue. It is interesting to study the synchronization of neurons and the development and transition of firing pattern in the coupled neurons in two-dimensional space. Up to date, the mechanism of instability[17–22] of the spiral wave has been investigated extensively, and many schemes have also been proposed to eliminate or remove the spiral wave in the reaction-diffusion system.[23–29] Furthermore, noise-induced transitions of patterns in the excitable media have attracted much attention, for example, noise can induce instability of spiral wave.[30] For simplicity, the Gaussian white noise is often used to study the transition of pattern in the excitable media and spatial-extended system. In fact, it is more practical to describe the effect of noise with the coloured noise by introducing the factor of correlation time instead of the Gaussian white noise.

There are many famous neuron models, and the Hindmarsh–Rose (HR) neuron[1] is investigated in this study. In our works, nearest-neighbour coupling in the HR neurons with appropriate coefficient and parameters are used to develop a stable rotating spiral wave and the initial values will be offered available. Then the factor of synchronization is defined and the Gaussian coloured noise will be introduced into all the neurons to study the breakup of spiral wave. The distribution for factor of synchronization and snapshots of membrane voltage will be plotted to discuss the breakup of spiral wave.

It is interesting and important to study the action and role of noise in the biological and neuronal system because the neurons often are sensitive to the external noise. It is more practical to simulate the uncertain fluctuation in the spatiotemporal system by introducing coloured noise into the system rather than the Gaussian white noise. The widely used coloured noise[31] $Q(t)$ is often produced from the Gaussian
white noise $\Gamma(t)$, and the statistical correlation is often described by
\begin{equation}
\langle Q(t) \rangle = 0, \quad \langle Q(t) Q(t') \rangle = \frac{D_0}{\tau} e^{-|t-t'|/\tau},
\end{equation}
where $D_0$ is the intensity and $\tau$ is the correlation time. It describes the case for Gaussian white noise at $\tau \to 0$. Clearly, the Gaussian coloured noise can be decided by the intensity and the correlation time. The activator is more sensitive than the inhibitor to the external noise, therefore, the coloured noise is added to the right side of the formula which describes the evolution of the activator. In this study, we introduce a Gaussian coloured noise into the coupled HR neurons and study the instability of spiral pattern with different intensities and correlation times, respectively. There is complex mutual coupling and interaction among the neurons, and it is important to define some statistical variables to measure the correlation of all the neurons or sites. It is convenient to calculate the amplitude error of few neurons. It could be acceptable to investigate the correlation of enormous neurons based on the mean field theory. Therefore, a new scale factor is proposed to study the instability of spiral wave even though the snapshots showing is available. A mean field $F$ and factor synchronization $R$ are defined to discuss the evolution of the voltage of all the neurons. The mean field $F$ and the factor of synchronization $R$ can be described by
\begin{align}
F &= \frac{1}{N^2} \sum_{j=1}^{N} \sum_{i=1}^{N} V_{ij} = \langle V \rangle_s, \\
R &= \frac{\langle F^2 \rangle - \langle F \rangle^2}{\frac{1}{N^2} \sum_{j=1}^{N} \sum_{i=1}^{N} (V_{ij}^2) - \langle V_{ij} \rangle^2},
\end{align}
where the variable $V_{ij}$ describes the activator of the system and $\langle V \rangle_s$ is the mean value of all neurons. Here $V_{ij}$ represents the variable $x_{ij}$ in the coupled HR neurons, which is given in the following. Instant $N^2$ is the total number of all neurons. In the case of complete synchronization of all the neurons, all the neurons will be the same amplitude and the error of each pair of neurons will equal to zero. As a result, the factor of synchronization is close to zero as well. In other words, the factor of synchronization will equal to zero when the whole media becomes homogenous, and all the neurons become complete synchronization. Furthermore, it just indicates that these neurons reach general synchronization when the factor of synchronization is above and beyond zero.

The three-variable HR neuron model of action potential has been proposed as a mathematical representation of the firing behaviour of neurons.\textsuperscript{1} Our aim is to study the instability of spiral wave in the coupled HR neurons with nearest-neighbour couplings\textsuperscript{[33–35]} by introducing Gaussian coloured noise into all the neurons. When the Gaussian coloured noise is imposed on the whole neuronal system, the coupled HR neurons with nearest-neighbour couplings are often described by
\begin{align}
dx_{i,j}/dt &= y_{i,j} - ax_{i,j}^3 + bx_{i,j}^2 - z_{i,j} + I \\
&\quad + D(x_{i+1,j} + x_{i-1,j} + x_{ij+1} + x_{ij-1} - 4x_{ij}) + Q(t), \\
dy_{i,j}/dt &= c - dx_{i,j}^2 - y_{i,j}, \\
dz_{i,j}/dt &= r(s(x_{i,j} + \chi) - z_{i,j}),
\end{align}
where the variable $x$ used to define the membrane action potential, variable $y$ is associated with the fast current $(Na^+)$ and considered as a recovery variable. The variable $z$ is a slow adaptation current maybe linked to $Ca^{2+}$. Parameters $a, b, c, d,$ and $s$ are often invariable, and parameter $r$ is a proportional constant, while parameter $\chi$ can be regarded as the reversal potential of $Ca^{2+}$, $I$ is the external current and $D$ is the coupling coefficient. $Q(t)$ is the Gaussian coloured noise and parameter $k$ is switched with 0 or 1 to close or open the noise to the neurons. These parameters are often selected with $a = 1.0, b = 3.0, c = 1.0, d = 5.0, s = 4.0, \chi = 1.6$ and different intensities of external current $I$ can induce single neuron to become chaotic or periodical and the parameters $r$ and $\chi$ are often treated as bifurcation parameters to analyse the transition of firing pattern for few neurons.

**Fig. 1.** Formation of spiral wave in Hindmarsh-Rose neuron. The snapshots of membrane voltage are shown in gray. Here $x(91:93, 1:100) = 2.0, y(91:93, 1:100) = 2.0, z(91:93, 1:100) = -1.0; x(94:96, 1:100) = 0.0, y(94:96, 1:100) = 0.0, z(94:96, 1:100) = 0.0; x(97:99, 1:100) = -1.0, y(97:99, 1:100) = -1.0, z(97:99, 1:100) = 2.0; and other neurons are selected with $x(i,j) = -1.31742, y(i,j) = -7.67799, z(i,j) = 1.13032$.

In our numerical simulation tests, no-flux boundary condition is used to find appropriate solutions for Eq. (1), the total number of neurons is $N^2 = 200 \times 200$ and time step $h = 0.02$ is used. A stable rotating spiral wave could be developed within 400 time units using
coupling coefficient $D = 0.5$ and $I = 1.315$ (or $D = 1$, $D_0 = 0.001$ and $\tau = 2.0$). A stable rotating spiral wave is plotted in Fig. 1.

The gaussian coloured noise with different intensities and correlation times is imposed on all the coupled HR neurons and the factor of synchronization and membrane voltages in different sites are calculated numerically. In Fig. 2 we show the evolution of synchronization of factor $R$ vs intensity of noise, and in Fig. 3, we illustrate the evolution of synchronization factor $R$ vs correlation time.

The curve in Fig. 2 shows that the factors of synchronization begin to increase with increasing intensity of coloured noise and some peaks could be found due to the stochastic resonance when the coloured noise works on all the coupled neurons.

The curve in Fig. 3 illustrates that the factors of synchronization begin to decrease with increasing correlation time and certain peaks and sudden turning points are also found when the coloured noise is imposed on all the neurons. Furthermore, the distribution for factor of synchronization vs intensity and correlation time of noise is plotted in Fig. 4.

Within the vicinity of dense region about intensity $D_0 = 0.03$, the factor of synchronization begins to change suddenly which indicates sudden transition of spiral wave, that is breakup of spiral wave. In comparison of the results in Figs. 2–4, it is confirmed that smaller correlation time and/or stronger intensity of gaussian coloured noise easier cause breakup of spiral wave, and the sudden changing point in the curve for
factor of synchronization just indicates the transition. Finally, the snapshots of membrane voltage in different sites with intensity \( D_0 = 0.03 \) for correlation time \( \tau = 3 \) and \( D_0 = 0.04 \) for correlation time \( \tau = 15 \) are plotted in Fig. 5, which shows the breakup of spiral wave. The snapshots in Fig. 5 confirm that breakup of spiral could be induced when the gaussian coloured noise is imposed on all the neurons. Smaller correlation or stronger intensity usually cause the breakup of spiral wave. The numerical results confirm that the breakup of spiral wave can occur when the intensity of coloured noise surpasses a upper threshold or the correlation time is decreased to a bottom threshold. Finally, it is important to discuss the mechanism of the breakup of the spiral wave in the coupled HR neurons. In this work, the Gaussian coloured noise is imposed on the right side of the formula which decides the evolution of activator \( x \). From the point of nonlinear dynamics, the inputting of colour noise just changes the external current as \( I = 1.315 + Q(t) \). As mention above, the single HR neuron could become chaotic with the increasing intensity of external current and it could be confirmed by calculating the spectrum of Lyapunov exponent. The coloured noise is imposed on each neuron, all the neurons become chaotic due to nearest-neighbour coupling when the gaussian coloured noise with appropriate intensity and/or correlation time is imposed on the coupled neurons. The excitable neurons response to the external stimuli from noise sensitively and it could encounter global instability synchronically, as a result, the breakup of spiral wave is observed.

In summary, we have reported the formation and breakup of spiral wave in the HR neurons with complete nearest-neighbour couplings. The factor of synchronization is defined to discuss the transition of spiral wave when the gaussian coloured noise in introduced into all the neurons. It is found that the spiral wave encounters breakup when the correlation time is decreased or the intensity of coloured noise is increased to a certain threshold. The factor of synchronization is also found to be increased with increasing intensity of coloured noise and/or decreasing correlation time and peaks are observed in the curve for factor of synchronization due to stochastic resonance. It is also confirmed that the first sudden changing point in the curve \( R - D_0 \) (factor of synchronization vs intensity of noise) indicates the threshold for breakup of spiral wave and it could be confirmed by observing the snapshots. It is also interesting and challenging to study the problem when the neurons are connected with other types, for example, small-world connection.\[^{36}\] and these interesting problems could be carried out in our forthcoming works in detail.

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References